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Transient elastic wave propagation in an infinite Timoshenko beam on viscoelastic foundation

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Abstract

This paper presents an effective numerical method for solving elastic wave propagation problems in an infinite Timoshenko beam on viscoelastic foundation in time domain. In order to use the finite element method to model the local complicated material properties of the infinite beam as well as foundation, two artificial boundaries are needed in the infinite system so as to truncate the infinite beam into a finite beam. This treatment requires an appropriate boundary condition derived and applied on the corresponding truncated boundaries. For this purpose, the time-dependent equilibrium equation of motion for beam is changed into a linear ordinary differential equation by using the operator splitting and the residual radiation methods. Simultaneously, an artificial parameter is employed in the derivation. As a result, the high-order accurate artificial boundary condition, which is local in time, is obtained by solving the ordinary differential equation. The numerical examples given in this paper demonstrate that the proposed method is of high accuracy in dealing with elastic wave propagation problems in an infinite foundation beam.

Keywords: Time-dependent; Elastic wave; Infinite domain; Artificial boundary conditions; Viscoelastic foundation; Timoshenko beam

1. Introduction

The infinite beam on elastic foundation has a sound background in civil engineering, such as railroad track system, highway and airfield pavements, and so forth. Considerable research has been conducted to investigate the response of infinite foundation beam subjected to dynamic loads. The Winkler model is usually adopted for foundation due to its simple form, while a relative realistic model is the two-parameter Winkler–Pasternak model (Feng and Cook, 1983). The Winkler foundation including material damping is another more practical model for dynamic loading cases. To model the infinite beam, there are two simplified beam theories: the Bernoulli–Euler and the Timoshenko theories (Hu, 1984). Including the one- or two-parameter model of foundation, the Bernoulli–Euler beam theory has usually been used to investigate wave propagation in an infinite elastic beam resting on various foundations. Due to its simpler

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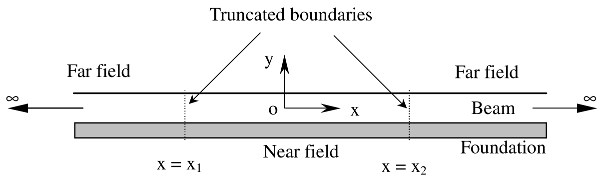
mathematical form, some analytical solutions are available for the homogeneous infinite beam on various foundations to dynamic loads (Kenney, 1954; Stadler and Shreeves, 1970; McGhie, 1990; Sun, 2001). However, only the flexural wave could propagate in the Bernoulli–Euler beam, whereas the shear wave would be ignored. This drawback limits the application of the Bernoulli–Euler beam theory only to motion where wavelength is larger compared to the height of the beam. Due to the wavelengths generated by most dynamic loads, especially the impulse loads, can be expected as the same order as lateral dimensions of the beam, it is almost necessary to use the Timoshenko beam theory in contrast to the Bernoulli–Euler theory to model the infinite beam on elastic foundation (Crandall, 1957; Achenbach and Sun, 1965; Wang and Gagnon, 1978; Felszeghy, 1996a,b; Billger and Folkow, 1998; Folkow et al., 1998; Chen et al., 2001). It is noted that almost all of the available closed-form solutions are derived on the technique of integral transforms. Therefore, only are the ideal homogeneous beam and foundation models considered in details. For the practical engineering problems, it is necessary to develop the numerical methods to simulate the infinite beam as well as foundation with inhomogeneous and complicated character.

Up to now, the numerical methods, such as finite difference and finite element method, are seldom used to investigate the wave propagation in the infinite beam on elastic foundation. The reason may be that these numerical methods are domain-dependent, that is, the computation domain must be finite. Due to the powerful ability of the finite element method to inhomogeneous materials and nonlinear problems, it is in urgent need to develop the time-domain numerical techniques based on the finite element method to model the wave propagation in infinite foundation beam. Wang et al. (1984) investigated the traveling wave in a Timoshenko beam on elastic foundation to dynamic loads by the method of direct analysis. The model used in the direct analysis method is to truncate the infinite beam into a finite beam with fixed boundary conditions on the truncated boundaries. This treatment requires that for the long-term response analysis of the concerned region, the length of the beam may become extensive long to prevent the response of the concerned region from being contaminated by the reflected waves on the truncated boundaries. To reduce the size of the computational domain, the truncated boundary should be taken near the concerned domain. For this purpose, it is necessary to develop and use the appropriate truncated boundary condition, which represents the effect of the truncated semi-infinite beam on viscoelastic foundation (Givoli, 1992). Since the local boundary conditions, which are local in time, space or in both time and space, involve simple expressions for artificial boundary, they usually result in a significant reduction in computer efforts. However, some spurious waves at the truncated boundary may degrade the accuracy and robustness of the numerical procedures available, especially when the truncated boundary is close to the dynamic source. If an exact boundary condition is applied on the truncated boundary, a high accurate numerical solution can be obtained and the total number of elements can be reduced significantly. However, the exact artificial boundary conditions, such as the mentioned closed-form solutions, are too complicated to be implemented numerically due to the involvement of special functions and convolution integrals (Tsynkov, 1998; Givoli, 1999). Recently, a new class of high-order accurate artificial boundary conditions has been proposed and studied (Grote and Keller, 1995, 1996; Hagstrom and Hariharan, 1998; Thompson and Huan, 1999; Huan and Thompson, 2000; Thompson et al., 2001; Givoli, 2001; Givoli and Patlashenko, 2002; Liu and Xu, 2002). Since the high-order artificial boundary conditions are local in time, nonlocal on the truncated boundaries for transient wave problems, they can be easily implemented into a conventional finite element method, without affecting the sparse structure of the stiffness and mass matrixes (Thompson and Huan, 2000). Up to now, the high-order accurate artificial boundary conditions are available for scalar wave propagations in two-dimensional or three-dimensional infinite domain. The time-dependent artificial boundary conditions for elastic wave problems are obtained only for layered domain (Kausel, 1994; Guddati and Tassoulas, 1999; Park and Tassoulas, 2002) and full three-dimensional domain (Grote and Keller, 2000). However, the time-dependent artificial boundary conditions for infinite beam on viscoelastic foundation are not forthcoming. Therefore, a major need exists for a high-order accurate time-dependent artificial boundary condition for infinite beam on viscoelastic foundation.

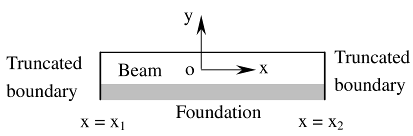
In this paper, a transient elastic wave propagation problem is investigated in an infinite Timoshenko beam on viscoelastic foundation. In order to use the finite element method to simulate the traveling wave in infinite beam, two artificial boundaries are introduced to truncate the original infinite beam into one finite and two semi-infinite beams. By using the operator splitting method, the time-dependent partial differential equation is changed into an ordinary differential equation. In addition, an artificial parameter with respect to elastic wave velocities is used to obtain the high-order accurate solution in a rigorously mathematical manner. The rest of the paper is arranged as follows. The governing partial differential equation and the corresponding initial-boundary conditions for an elastic wave propagation problem in an infinite Timoshenko beam on viscoelastic foundation are presented first and followed by the operator splitting method and auxiliary functions used to derive the high-order accurate artificial boundary conditions on the truncated boundaries of a semi-infinite foundation beam. And then the implementation of the obtained artificial boundary conditions in the finite element analysis is discussed. In the numerical example section, the accuracy and convergence of the proposed method are examined by two examples. Some discussions and conclusions are drawn in the final section.

2. Problem statement

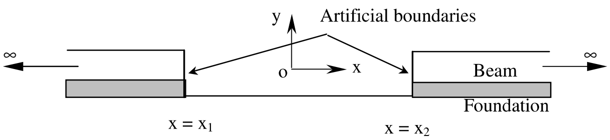
An infinite elastic beam on viscoelastic foundation is considered in this paper, whose motion is assumed to be in-plane (shown in Fig. 1). The infinite beam is taken as Timoshenko beam, while the viscoelastic foundation beneath the infinite beam is modeled by the continuous springs and dashpots uniformly distributed along the beam length. The infinite beam as well as foundation is divided into one finite and two semi-infinite regions by introducing two artificial boundaries, $x = x_1$ and $x = x_2$, respectively. When subjected to a dynamic load, the finite beam on viscoelastic foundation which may have inhomogeneous and



(a) Infinite foundation beam



(b) Artificial finite foundation beam



(c) Semi-infinite foundation beams

Fig. 1. Infinite beam on viscoelastic foundation models.

complicated material properties, is simulated by the finite element method, while the semi-infinite beam on viscoelastic foundation without direct loading excitation, which is homogeneous material, can be simulated using the high-order accurate boundary condition. The main objective of this study is to determine the high-order accurate boundary condition, which is a relationship between displacement and resultant on the truncated boundary. For this purpose, it is necessary to consider the elastic wave propagation problem in semi-infinite beam on viscoelastic foundation. Due to the similarity between these two semi-infinite models, we will focus on the derivation of the high-order accurate boundary condition on the truncated boundary of $x = x_2$ below. The forces and the deformations of an infinitesimal beam element on viscoelastic foundation are shown in Fig. 2. The motion equilibrium can be given by (Hu, 1984)

$$\frac{\partial Q}{\partial x} = -\rho A\ddot{v} - d\dot{v} - kv \tag{1}$$

$$\frac{\partial M}{\partial x} = Q + \rho J\ddot{\theta} - d_1\dot{\theta} - k_1\theta \tag{2}$$

where v and θ represent the beam deformation and the bending rotation, respectively; ρ, A, J, d, d_1, k and k_1 represent the mass density, the area of the cross-section, the second moment of area of the beam section, the viscous-damping coefficients, the spring constants, respectively; Q and M represent the shear force and the bending moment, respectively.

The shear force Q and the bending moment M can be expressed by the corresponding deformations as

$$Q(x,t) = \mu AG\beta \tag{3}$$

$$M(x,t) = EJ\frac{\partial \theta}{\partial x} \tag{4}$$

where β represents the shear angle, μ shear factor, and E and G the Young's and shear moduli of the beam, respectively.

The relationship between the bending and shear angles is given by

$$\theta = \frac{\partial v}{\partial x} + \beta \tag{5}$$

Substitutions of Eqs. (3)–(5) into Eqs. (1) and (2), respectively, yield

$$\mu AG\frac{\partial^2 v}{\partial x^2} - \mu AG\frac{\partial \theta}{\partial x} - kv = \rho A\ddot{v} + d\dot{v} \tag{6}$$

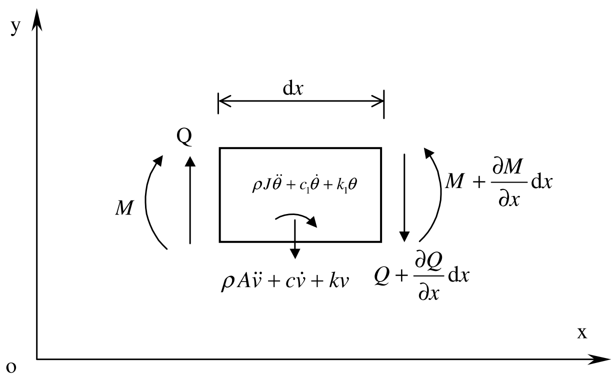


Fig. 2. Forces of an infinitesimal foundation beam element.

$$EJ\frac{\partial^2\theta}{\partial x^2} + \mu AG\frac{\partial v}{\partial x} - (\mu AG + k_1)\theta = \rho J\ddot{\theta} + d_1\dot{\theta} \tag{7}$$

Written in the matrix form, Eqs. (6) and (7) can be expressed as

$$\mathbf{E}_0\frac{\partial^2\mathbf{u}}{\partial x^2} + (\mathbf{E}_1^{\mathrm{T}} - \mathbf{E}_1)\frac{\partial\mathbf{u}}{\partial x} - \mathbf{E}_2\mathbf{u} = \mathbf{M}_0\frac{\partial^2\mathbf{u}}{\partial t^2} + \mathbf{E}_3\frac{\partial\mathbf{u}}{\partial t} \tag{8}$$

where

$$\mathbf{u} = [v \quad \theta]^{\mathrm{T}} \tag{9a}$$

$$\mathbf{E}_0 = \begin{bmatrix} \mu GA & 0 \\ 0 & EJ \end{bmatrix} \tag{9b}$$

$$\mathbf{E}_1 = \begin{bmatrix} 0 & 0 \\ -\mu GA & 0 \end{bmatrix} \tag{9c}$$

$$\mathbf{E}_2 = \begin{bmatrix} k & 0 \\ 0 & \mu GA + k_1 \end{bmatrix} \tag{9d}$$

$$\mathbf{M}_0 = \begin{bmatrix} \rho A & 0 \\ 0 & \rho J \end{bmatrix} \tag{9e}$$

$$\mathbf{E}_3 = \begin{bmatrix} d & 0 \\ 0 & d_1 \end{bmatrix} \tag{9f}$$

Eq. (8) turns out to be the equation that applies to the semi-infinite beam on viscoelastic foundation. It is noted that both $\mathbf{E}_0$ and $\mathbf{M}_0$ are diagonal and positive definite matrixes.

The motion of the semi-infinite beam on viscoelastic foundation is assumed to start from rest. Therefore, the initial conditions are

$$\mathbf{u}(x,0) = \frac{\partial\mathbf{u}}{\partial t}(x,0) = 0 \quad (x \geqslant x_2) \tag{10}$$

The displacement and velocity on the boundary of the semi-infinite beam are, respectively,

$$\mathbf{u}(x_2,t) = \mathbf{u}_0(t) \tag{11}$$

$$\frac{\partial\mathbf{u}(x_2,t)}{\partial t} = \dot{\mathbf{u}}_0(t) \tag{12}$$

where $\mathbf{u}_0(t)$ and $\dot{\mathbf{u}}_0(t)$ are the transient responses of the finite beam region on the truncated boundary obtained by the finite element method.

Now, the resultant forces corresponding to all the degrees of freedom of the boundary are addressed. Recalling Eqs. (3)–(5), the resultant force vector on the boundary is written as

$$\mathbf{R} = -\mathbf{E}_0\frac{\partial\mathbf{u}}{\partial x} - \mathbf{E}_1^{\mathrm{T}}\mathbf{u} \tag{13}$$

For the elastic wave propagation in a semi-infinite beam on viscoelastic foundation, a relationship among displacement $\mathbf{u}_0(t)$, velocity $\dot{\mathbf{u}}_0(t)$ and resultant force $\mathbf{R}(t)$ on the boundary is needed. This requires the derivative, $\partial\mathbf{u}/\partial x$ in Eq. (13), which must be represented by the corresponding responses of semi-infinite beam on truncated boundary.

3. The residual radiation method

The employment of the residual radiation method (Liu and Xu, 2002) is to transfer Eq. (8) into a linear first-order ordinary differential equation by factorizing the partial differential equation and introducing a set of residual radiation functions. Due to the positive definition of $\mathbf{E}_0$ and $\mathbf{M}_0$, Eq. (8) can be rewritten as

$$\frac{\partial^2 \mathbf{v}_0}{\partial x^2} - \mathbf{\Lambda}^2 \frac{\partial^2 \mathbf{v}_0}{\partial t^2} = \mathbf{E}\mathbf{v}_0 + \mathbf{D}\frac{\partial \mathbf{v}_0}{\partial x} + \mathbf{F}\frac{\partial \mathbf{v}_0}{\partial t} \tag{14}$$

where

$$\mathbf{v}_0 = \mathbf{N}\mathbf{u} \tag{15a}$$

$$\mathbf{N} = \mathrm{diag}(\sqrt{\mu GA}, \sqrt{EJ}) \tag{15b}$$

$$\mathbf{\Lambda}^2 = \mathrm{diag}\left(\frac{\rho}{\mu G}, \frac{\rho}{E}\right) \tag{15c}$$

$$\mathbf{E} = \mathrm{diag}\left(\frac{k}{\mu GA}, \frac{\mu GA + k_1}{EJ}\right) \tag{15d}$$

$$\mathbf{D} = \begin{bmatrix} 0 & \sqrt{\mu GA/EJ} \\ -\sqrt{\mu GA/EJ} & 0 \end{bmatrix} \tag{15e}$$

$$\mathbf{F} = \mathrm{diag}\left(\frac{d}{\mu GA}, \frac{d_1}{EJ}\right) \tag{15f}$$

Obviously, the left-hand side of Eq. (14) can be factorized as follows:

$$\left(\frac{\partial}{\partial x} - \mathbf{\Lambda}\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x} + \mathbf{\Lambda}\frac{\partial}{\partial t}\right)\mathbf{v}_0 = \mathbf{E}\mathbf{v}_0 + \mathbf{D}\frac{\partial \mathbf{v}_0}{\partial x} + \mathbf{F}\frac{\partial \mathbf{v}_0}{\partial t} \tag{16}$$

The partial differential operators

$$\frac{\partial}{\partial x} - \mathbf{\Lambda}\frac{\partial}{\partial t} \quad \text{and} \quad \frac{\partial}{\partial x} + \mathbf{\Lambda}\frac{\partial}{\partial t}$$

in Eq. (16) represent the incoming and outgoing waves, respectively. Introducing an auxiliary vector $\mathbf{v}_1$ into Eq. (16) yields the following equations:

$$\frac{\partial \mathbf{v}_0}{\partial x} + \mathbf{\Lambda}\frac{\partial \mathbf{v}_0}{\partial t} = \mathbf{v}_1 \tag{17}$$

$$\frac{\partial \mathbf{v}_1}{\partial x} - \mathbf{\Lambda}\frac{\partial \mathbf{v}_1}{\partial t} = \mathbf{E}\mathbf{v}_0 + \mathbf{D}\mathbf{v}_1 - \mathbf{H}\mathbf{v}_0^{-1} \tag{18}$$

where

$$\mathbf{H} = \mathbf{D}\mathbf{\Lambda} - \mathbf{F} \tag{19}$$

$$\mathbf{v}_i^{-j} = \frac{\partial^j \mathbf{v}_i}{\partial t^j} \quad (j > 0,\ i = 0, 1, 2, \ldots) \tag{20}$$

It is clear that Eq. (17) is similar to the radiation boundary conditions. If the function $\mathbf{v}_0$ satisfies the standard wave equation, i.e., $\mathbf{D} = 0$, $\mathbf{E} = 0$ and $\mathbf{H} = 0$, then $\mathbf{v}_1 = 0$, so that all the outgoing waves will travel through the artificial boundary without any residual. However, for elastic wave propagation in semi-infinite beam, matrixes $\mathbf{D}$, $\mathbf{E}$ and $\mathbf{H}$ do not vanish, the auxiliary function $\mathbf{v}_1$ exists in the systems, which represents the residual radiation of function $\mathbf{v}_0$.

To obtain a higher-order accurate numerical solution, it is important to determine the first-order residual function $\mathbf{v}_1$ accurately. Followed is an efficient method proposed to determine the lower residual functions through the higher ones. As an example, the second order of radiation vector $\mathbf{v}_2$ is introduced to obtain $\mathbf{v}_1$ as follows:

$$\frac{\partial \mathbf{v}_1}{\partial x} + \mathbf{\Sigma}\frac{\partial \mathbf{v}_1}{\partial t} = \mathbf{v}_2 \tag{21}$$

where $\mathbf{\Sigma} = \sigma\mathbf{I}_2$ is an introduced diagonal matrix, $\mathbf{I}_2$ is a 2-order identity matrix and parameter σ is employed to reduce the radiation residual, and can be evaluated $\sigma = \frac{1}{2}(\Lambda_1 + \Lambda_2)$ for instance. Application of the incoming partial differential operator $(\partial/\partial x) - \mathbf{\Lambda}(\partial/\partial t)$ to Eq. (21) yields

$$\left(\frac{\partial}{\partial x} - \mathbf{\Lambda}\frac{\partial}{\partial t}\right)\mathbf{v}_2 = \left(\frac{\partial}{\partial x} - \mathbf{\Lambda}\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x} + \mathbf{\Sigma}\frac{\partial}{\partial t}\right)\mathbf{v}_1 \tag{22}$$

Substitution of Eq. (18) into Eq. (22) with exchange of operators yields

$$\left(\frac{\partial}{\partial x} - \mathbf{\Lambda}\frac{\partial}{\partial t}\right)\mathbf{v}_2 = \left(\frac{\partial}{\partial x} + \mathbf{\Sigma}\frac{\partial}{\partial t}\right)\left(\mathbf{E}\mathbf{v}_0 + \mathbf{D}\mathbf{v}_1 - \mathbf{H}\mathbf{v}_0^{-1}\right) \tag{23}$$

Obviously, the matrix $\mathbf{\Sigma}$ is an exchangeable matrix, that is, $\mathbf{\Sigma E} = \mathbf{E\Sigma}$, $\mathbf{\Sigma D} = \mathbf{D\Sigma}$ and so forth. Hence, accounting for Eq. (17), the first term in the right-hand side of Eq. (23) can be rewritten as

$$\left(\frac{\partial}{\partial x} + \mathbf{\Sigma}\frac{\partial}{\partial t}\right)(\mathbf{E}\mathbf{v}_0) = \mathbf{E}\mathbf{v}_1 + \mathbf{E}(\mathbf{\Sigma} - \mathbf{\Lambda})\mathbf{v}_0^{-1} \tag{24}$$

Likewise, the third term in the right-hand side of Eq. (23) yields

$$\left(\frac{\partial}{\partial x} + \mathbf{\Sigma}\frac{\partial}{\partial t}\right)(-\mathbf{H}\mathbf{v}_0^{-1}) = -\mathbf{H}\mathbf{v}_1^{-1} - \mathbf{H}(\mathbf{\Sigma} - \mathbf{\Lambda})\mathbf{v}_0^{-2} \tag{25}$$

Therefore, a pair of partial differential equations are obtained as

$$\frac{\partial \mathbf{v}_1}{\partial x} + \mathbf{\Sigma}\frac{\partial \mathbf{v}_1}{\partial t} = \mathbf{v}_2 \tag{26}$$

$$\frac{\partial \mathbf{v}_2}{\partial x} - \mathbf{\Lambda}\frac{\partial \mathbf{v}_2}{\partial t} = \mathbf{E}\mathbf{v}_1 + \mathbf{E}(\mathbf{\Sigma} - \mathbf{\Lambda})\mathbf{v}_0^{-1} + \mathbf{D}\mathbf{v}_2 - \mathbf{H}\mathbf{v}_1^{-1} - \mathbf{H}(\mathbf{\Sigma} - \mathbf{\Lambda})\mathbf{v}_0^{-2} \tag{27}$$

In the same way, the following equations with respect to the third residual radiation function $\mathbf{v}_3$ are obtained:

$$\frac{\partial \mathbf{v}_2}{\partial x} + \mathbf{\Sigma}\frac{\partial \mathbf{v}_2}{\partial t} = \mathbf{v}_3 \tag{28}$$

$$\frac{\partial \mathbf{v}_3}{\partial x} - \mathbf{\Lambda}\frac{\partial \mathbf{v}_3}{\partial t} = \mathbf{E}\mathbf{v}_2 + \mathbf{E}(\mathbf{\Sigma} - \mathbf{\Lambda})\mathbf{v}_1^{-1} + \mathbf{E}(\mathbf{\Sigma} - \mathbf{\Lambda})^2\mathbf{v}_0^{-2} + \mathbf{D}\mathbf{v}_3 - \mathbf{H}\mathbf{v}_2^{-1} - \mathbf{H}(\mathbf{\Sigma} - \mathbf{\Lambda})\mathbf{v}_1^{-2} - \mathbf{H}(\mathbf{\Sigma} - \mathbf{\Lambda})^2\mathbf{v}_0^{-3} \tag{29}$$

Likewise, the following general factorizing forms with respect to the $(n+1)$th residual radiation $\mathbf{v}_{n+1}$ can be obtained:

$$\frac{\partial \mathbf{v}_n}{\partial x} + \mathbf{\Sigma}\frac{\partial \mathbf{v}_n}{\partial t} = \mathbf{v}_{n+1} \quad (n = 1, 2, \ldots, N) \tag{30}$$

$$\frac{\partial \mathbf{v}_{n+1}}{\partial x} - \mathbf{\Lambda}\frac{\partial \mathbf{v}_{n+1}}{\partial t} = \mathbf{E}\sum_{i=0}^{n}\left((\mathbf{\Sigma} - \mathbf{\Lambda})^i \mathbf{v}_{n-i}^{-i}\right) - \mathbf{H}\sum_{i=0}^{n}\left((\mathbf{\Sigma} - \mathbf{\Lambda})^i \mathbf{v}_{n-i}^{-i-1}\right) + \mathbf{D}\mathbf{v}_{n+1} \quad (n = 1, 2, \ldots, N) \tag{31}$$

where N is the highest order of the auxiliary function. The correctness of Eqs. (30) and (31) can be proven using the inductive method.

Still is a difficulty to implement the above factorizing partial differential equations into any numerical procedure due to the presentation of the spatial derivatives and the high-order temporal derivatives. The spatial derivatives can be eliminated from the above equations by subtracting Eq. (18) from Eq. (21) as follows:

$$(\mathbf{\Sigma} + \mathbf{\Lambda})\frac{\partial \mathbf{v}_1}{\partial t} = \mathbf{v}_2 - \mathbf{E}\mathbf{v}_0 - \mathbf{D}\mathbf{v}_1 + \mathbf{H}\mathbf{v}_0^{-1} \tag{32}$$

Similarly, the second ordinary differential equation can be obtained as

$$(\mathbf{\Sigma} + \mathbf{\Lambda})\frac{\partial \mathbf{v}_2}{\partial t} = \mathbf{v}_3 - \mathbf{E}\mathbf{v}_1 - \mathbf{E}(\mathbf{\Sigma} - \mathbf{\Lambda})\mathbf{v}_0^{-1} - \mathbf{D}\mathbf{v}_2 + \mathbf{H}\mathbf{v}_1^{-1} + \mathbf{H}(\mathbf{\Sigma} - \mathbf{\Lambda})\mathbf{v}_0^{-2} \tag{33}$$

Likewise, a general form of the ordinary differential equation can be written as

$$(\mathbf{\Sigma} + \mathbf{\Lambda})\frac{\partial \mathbf{v}_n}{\partial t} = \mathbf{v}_{n+1} - \mathbf{E}\sum_{i=0}^{n-1}\left((\mathbf{\Sigma} - \mathbf{\Lambda})^i \mathbf{v}_{n-1-i}^{-i}\right) + \mathbf{H}\sum_{i=0}^{n-1}\left((\mathbf{\Sigma} - \mathbf{\Lambda})^i \mathbf{v}_{n-1-i}^{-i-1}\right) - \mathbf{D}\mathbf{v}_n \quad (n = 1, 2, \ldots, N) \tag{34}$$

To reduce the high-order temporal derivatives in the above ordinary differential equations, the notation $\mathbf{v}_i^{-j}$ in Eq. (20) is extended as

$$\mathbf{v}_i^j = \int_0^{\tau_j} \cdots \int_0^{\tau_2}\int_0^{\tau_1} \mathbf{v}_i \, \mathrm{d}\tau_1 \, \mathrm{d}\tau_2 \cdots \mathrm{d}\tau_j \quad (j \geqslant 0, \; i = 0, 1, 2, 3, \ldots) \tag{35}$$

Using the above notions, Eq. (32) can be rewritten as

$$(\mathbf{\Sigma} + \mathbf{\Lambda})\frac{\partial \mathbf{v}_1}{\partial t} - \frac{\partial \mathbf{v}_2^1}{\partial t} = -\mathbf{E}\mathbf{v}_0 - \mathbf{D}\mathbf{v}_1 + \mathbf{H}\mathbf{v}_0^{-1} \tag{36}$$

Integrating Eq. (33) with respect to time once, the second ordinary differential equation is expressed as

$$(\mathbf{\Sigma} + \mathbf{\Lambda})\frac{\partial \mathbf{v}_2^1}{\partial t} - \frac{\partial \mathbf{v}_3^2}{\partial t} = -\mathbf{E}\mathbf{v}_1^1 - \mathbf{E}(\mathbf{\Sigma} - \mathbf{\Lambda})\mathbf{v}_0 - \mathbf{D}\mathbf{v}_2^1 + \mathbf{H}\mathbf{v}_1 + \mathbf{H}(\mathbf{\Sigma} - \mathbf{\Lambda})\mathbf{v}_0^{-1} \tag{37}$$

Likewise, integrating the nth Eq. (34) with respect to time $n-1$ times leads to the follows:

$$\begin{aligned}(\mathbf{\Sigma} + \mathbf{\Lambda})&\frac{\partial \mathbf{v}_n^{n-1}}{\partial t} - \frac{\partial \mathbf{v}_{n+1}^n}{\partial t} \\ &= -\mathbf{E}\sum_{i=0}^{n-1}\left((\mathbf{\Sigma} - \mathbf{\Lambda})^i \mathbf{v}_{n-1-i}^{n-1-i}\right) + \mathbf{H}\sum_{i=0}^{n-1}\left((\mathbf{\Sigma} - \mathbf{\Lambda})^i \mathbf{v}_{n-1-i}^{n-i-2}\right) - \mathbf{D}\mathbf{v}_n^{n-1} \quad (n = 1, 2, \ldots, N)\end{aligned} \tag{38}$$

Eqs. (36)–(38) are all first-order temporal ordinary differential equations for the related residual radiation functions. The unknown functions are $\mathbf{v}_1, \mathbf{v}_2^1, \mathbf{v}_3^2, \ldots, \mathbf{v}_n^{n-1}$ and $\mathbf{v}_1^1, \mathbf{v}_2^2, \mathbf{v}_3^3, \ldots, \mathbf{v}_n^n$, while the known conditions are $\mathbf{v}_0$ and $\dot{\mathbf{v}}_0$, which are expressed in Eqs. (10)–(12), on the truncated boundary $x = x_2$. Hence, Eqs. (36)–(38) can be rewritten in the matrix form as

$$\begin{bmatrix} \mathbf{A}_0 & 0 \\ 0 & \mathbf{I} \end{bmatrix} \begin{Bmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \end{Bmatrix}_t = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{I} & 0 \end{bmatrix} \begin{Bmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \end{Bmatrix} + \begin{Bmatrix} \mathbf{q} \\ 0 \end{Bmatrix} \tag{39}$$

where $\mathbf{w}_1 = [\mathbf{v}_1 \quad \mathbf{v}_2^1 \quad \mathbf{v}_3^2 \quad \ldots \quad \mathbf{v}_n^{n-1}]^{\mathrm{T}}$ and $\mathbf{w}_2 = [\mathbf{v}_1^1 \quad \mathbf{v}_2^2 \quad \mathbf{v}_3^3 \quad \ldots \quad \mathbf{v}_n^n]^{\mathrm{T}}$. Matrixes $\mathbf{A}_0$, $\mathbf{A}_1$, $\mathbf{A}_2$ and vector $\mathbf{q}$ have the following block forms, respectively:

$$\mathbf{A}_0 = \begin{bmatrix} \boldsymbol{\Sigma}+\boldsymbol{\Lambda} & -\mathbf{I} & & & \\ & \boldsymbol{\Sigma}+\boldsymbol{\Lambda} & -\mathbf{I} & & \\ & & \boldsymbol{\Sigma}+\boldsymbol{\Lambda} & -\mathbf{I} & \\ & & \cdot & \cdot & \cdot \\ & & & & \boldsymbol{\Sigma}+\boldsymbol{\Lambda} \end{bmatrix}_{n\times n} \tag{40a}$$

$$\mathbf{A}_1 = \begin{bmatrix} -\mathbf{D} & & & & & & & \\ \mathbf{H}(\boldsymbol{\Sigma}-\boldsymbol{\Lambda}) & -\mathbf{D} & & & & & & \\ \mathbf{H}(\boldsymbol{\Sigma}-\boldsymbol{\Lambda})^2 & \mathbf{H}(\boldsymbol{\Sigma}-\boldsymbol{\Lambda}) & -\mathbf{D} & & & & & \\ \mathbf{H}(\boldsymbol{\Sigma}-\boldsymbol{\Lambda})^3 & \mathbf{H}(\boldsymbol{\Sigma}-\boldsymbol{\Lambda})^2 & \mathbf{H}(\boldsymbol{\Sigma}-\boldsymbol{\Lambda}) & -\mathbf{D} & & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & \cdots & & & & \\ \mathbf{H}(\boldsymbol{\Sigma}-\boldsymbol{\Lambda})^{n-3} & \mathbf{H}(\boldsymbol{\Sigma}-\boldsymbol{\Lambda})^{n-4} & & \cdots & \mathbf{H}(\boldsymbol{\Sigma}-\boldsymbol{\Lambda}) & -\mathbf{D} & & \\ \mathbf{H}(\boldsymbol{\Sigma}-\boldsymbol{\Lambda})^{n-2} & \mathbf{H}(\boldsymbol{\Sigma}-\boldsymbol{\Lambda})^{n-3} & & \cdots & \mathbf{H}(\boldsymbol{\Sigma}-\boldsymbol{\Lambda})^2 & \mathbf{H}(\boldsymbol{\Sigma}-\boldsymbol{\Lambda}) & -\mathbf{D} & \end{bmatrix}_{n\times n} \tag{40b}$$

$$\mathbf{A}_2 = \begin{bmatrix} \mathbf{0} & & & & & & & \\ -\mathbf{E} & \mathbf{0} & & & & & & \\ \mathbf{E}(\boldsymbol{\Sigma}-\boldsymbol{\Lambda})^1 & -\mathbf{E} & \mathbf{0} & & & & & \\ \mathbf{E}(\boldsymbol{\Sigma}-\boldsymbol{\Lambda})^2 & \mathbf{E}(\boldsymbol{\Sigma}-\boldsymbol{\Lambda}) & -\mathbf{E} & \mathbf{0} & & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & \cdots & & & & \\ \mathbf{E}(\boldsymbol{\Sigma}-\boldsymbol{\Lambda})^{n-4} & \mathbf{E}(\boldsymbol{\Sigma}-\boldsymbol{\Lambda})^{n-5} & & \cdots & \mathbf{E}(\boldsymbol{\Sigma}-\boldsymbol{\Lambda}) & -\mathbf{E} & \mathbf{0} & \\ \mathbf{E}(\boldsymbol{\Sigma}-\boldsymbol{\Lambda})^{n-3} & \mathbf{E}(\boldsymbol{\Sigma}-\boldsymbol{\Lambda})^{n-4} & & \cdots & \mathbf{E}(\boldsymbol{\Sigma}-\boldsymbol{\Lambda})^2 & \mathbf{E}(\boldsymbol{\Sigma}-\boldsymbol{\Lambda})^1 & -\mathbf{E} & \mathbf{0} \end{bmatrix}_{n\times n} \tag{40c}$$

$$\mathbf{q} = \begin{bmatrix} -\mathbf{E}\mathbf{v}_0 + \mathbf{H}\dot{\mathbf{v}}_0 \\ -\mathbf{E}(\boldsymbol{\Sigma}-\boldsymbol{\Lambda})\mathbf{v}_0 + \mathbf{H}(\boldsymbol{\Sigma}-\boldsymbol{\Lambda})\dot{\mathbf{v}}_0 \\ -\mathbf{E}(\boldsymbol{\Sigma}-\boldsymbol{\Lambda})^2\mathbf{v}_0 + \mathbf{H}(\boldsymbol{\Sigma}-\boldsymbol{\Lambda})^2\dot{\mathbf{v}}_0 \\ \cdots \\ -\mathbf{E}(\boldsymbol{\Sigma}-\boldsymbol{\Lambda})^{n-1}\mathbf{v}_0 + \mathbf{H}(\boldsymbol{\Sigma}-\boldsymbol{\Lambda})^{n-1}\dot{\mathbf{v}}_0 \end{bmatrix}_n \tag{40d}$$

Furthermore, the linear system (39) can be rewritten as the following standard ordinary differential equation:

$$\frac{\mathrm{d}\mathbf{w}}{\mathrm{d}t} = \mathbf{A}\mathbf{w} + \mathbf{w}^0 \tag{41}$$

where

$$\mathbf{w} = [\,\mathbf{w}_1 \quad \mathbf{w}_2\,]_{2n}^{\mathrm{T}} \tag{42a}$$

$$\mathbf{w}^0 = \left[\,\mathbf{A}_0^{-1}\mathbf{q} \quad \mathbf{0}\,\right]_{2n}^{\mathrm{T}} \tag{42b}$$

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_0^{-1}\mathbf{A}_1 & \mathbf{A}_0^{-1}\mathbf{A}_2 \\ \mathbf{I} & \mathbf{0} \end{bmatrix}_{2n\times 2n} \tag{42c}$$

From Eq. (10), the initial condition for $\mathbf{w}$ is obtained as

$$\mathbf{w} = [\mathbf{0} \quad \mathbf{0} \quad \cdots \quad \mathbf{0}]_{2n}^{\mathrm{T}} \tag{43}$$

Using the explicit second-order Adams–Bashforth algorithm (James et al., 1993), the numerical solution to Eq. (41) can be expressed as follows:

$$\mathbf{w}(t_{k+1}) = \mathbf{w}(t_k) + \frac{\Delta t}{2}[3(\mathbf{A}\mathbf{w}(t_k) + \mathbf{w}^0(t_k)) - (\mathbf{A}\mathbf{w}(t_{k-1}) + \mathbf{w}^0(t_{k-1}))] \tag{44}$$

where Δt is the time step, k is the number of time step and $t_k = k\Delta t$.

Once the residual radiation functions are solved from Eq. (44), then Eq. (17) can be rewritten as

$$\frac{\partial \mathbf{v}_0}{\partial x}(t_k) = -\mathbf{\Lambda}\frac{\partial \mathbf{v}_0}{\partial t}(t_k) + \mathbf{v}_1(t_k) \tag{45}$$

Multiplying the above expression by matrix $\mathbf{N}$ and substituting it into Eq. (13), the force at the boundary of semi-infinite foundation beam can be obtained as follows:

$$\mathbf{R}(t_k) = \mathbf{C}_0\frac{\partial \mathbf{u}}{\partial t}(t_k) - \mathbf{E}_1^{\mathrm{T}}\mathbf{u}(t_k) - \mathbf{N}\mathbf{v}_1(t_k) \tag{46}$$

where $\mathbf{C}_0 = \mathbf{N}\mathbf{\Lambda}\mathbf{N}^{\mathrm{T}}$ is a symmetric and positive definite matrix. Eq. (46) is the relationship between the force and displacement at the boundary, i.e., the discrete artificial boundary conditions for transient elastic wave propagation in the semi-infinite beam on viscoelastic foundation. Obviously, the above boundary condition is local in time since it only contains the displacement and velocity during two previous successive time instants (see Eq. (44)).

If only the first term in the right-hand side of Eq. (46) is considered, the viscous damping boundary condition is obtained as follows:

$$\mathbf{R}(t_k) = \mathbf{C}_0\frac{\partial \mathbf{u}}{\partial t}(t_k) \tag{47}$$

4. Finite element formulation

The proposed high-order accurate artificial boundary condition can be straightforwardly implemented in the finite element analysis (Thompson and Huan, 2000). Traditionally, the finite element discretized formulation of the Timoshenko beam on viscoelastic foundation can be expressed as

$$\mathbf{M}\ddot{\mathbf{y}} + \mathbf{C}\dot{\mathbf{y}} + \mathbf{K}\mathbf{y} = \mathbf{f} \tag{48}$$

where $\mathbf{M}$ is the global mass matrix, $\mathbf{C}$ the global viscous and radial damping matrix, $\mathbf{K}$ the global stiffness matrix, and $\mathbf{f}$ the global force vector due to the artificial boundary and other contributions. $\ddot{\mathbf{y}}$, $\dot{\mathbf{y}}$ and $\mathbf{y}$ are the global acceleration, velocity and displacement vectors, respectively. These global property matrixes and vectors can be obtained by assembling all of the element matrixes (Przemieniecki, 1968) and vectors including those of the artificial boundary conditions. The element property matrixes can be expressed as

$$\mathbf{M}^{\mathrm{e}} = \mathrm{diag}(\,\rho Al/2 \quad \rho Jl/2 \quad \rho Al/2 \quad \rho Jl/2\,) \tag{49a}$$

$$\mathbf{C}_{\mathrm{f}}^{\mathrm{e}} = \mathrm{diag}(\,dl/2 \quad d_1 l/2 \quad dl/2 \quad d_1 l/2\,) \tag{49b}$$

$$\mathbf{K}_{\mathrm{b}}^{\mathrm{e}} = \frac{EJ}{l^3(1+\Phi)}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & (4+\Phi)l^2 & -6l & (2-\Phi)l^2 \\ -12 & -6l & 12 & -6l \\ 6l & (2-\Phi)l^2 & -6l & (4+\Phi)l^2 \end{bmatrix} \tag{49c}$$

$$\mathbf{K}_\mathrm{f}^\mathrm{e} = \mathrm{diag}(\,kl/2 \quad k_1 l/2 \quad kl/2 \quad k_1 l/2\,) \tag{49d}$$

where l is the length of a beam element and $\Phi = (12EJ)/(\mu GAl^2)$, $\mathbf{C}_\mathrm{f}^\mathrm{e}$ is the viscous damping matrix of a foundation element, $\mathbf{K}_\mathrm{b}^\mathrm{e}$ and $\mathbf{K}_\mathrm{f}^\mathrm{e}$ are the stiffness matrixes of a Timoshenko beam element and a elastic foundation element, respectively.

Due to that the high-order accurate boundary condition needs both the displacements and velocities on the truncated boundary, the dynamic equation (48) is rewritten in a form of state space as follows:

$$\begin{bmatrix} \dot{\mathbf{y}} \\ \ddot{\mathbf{y}} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \dot{\mathbf{y}} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{M}^{-1}\mathbf{f} \end{bmatrix} \tag{50}$$

This linear system, Eq. (50), can be solved by the same numerical method as Eq. (41). Furthermore, Eq. (50) may be decoupled with Eq. (41) on the artificial boundary, it is possible to solve Eqs. (41) and (50) in an alternative manner, until the final time instant of interest is reached in the finite element analysis.

5. Numerical examples

The performance of the proposed high-order accurate artificial boundary condition is illustrated through some examples for time-dependent elastic wave propagation problems in infinite and semi-infinite beam on viscoelastic foundation. The first one is a semi-infinite homogeneous beam overlying on the viscoelastic foundation, and the second is a semi-infinite beam on the viscoelastic foundation with partly inhomogeneous properties. Both examples are performed in time domain in order to illustrate the efficiency of the present method. The exact solutions used for comparison are obtained from the consistency boundary method (Kausel et al., 1975).

5.1. A semi-infinite homogeneous beam on viscoelastic foundation

The transient response of a homogeneous beam on viscoelastic foundation (as shown in Fig. 3) is analyzed using the proposed high-order accurate nonreflecting boundary condition. Followed is the material parameters, $A = 0.25 \times 1.0$ m^2, $I = 0.02083$ m^4, $E = 4.29 \times 10^5$ kN m^2, $\gamma = 0.2$, $\rho = 2.35 \times 10^3$ kg/m^3, $k_0 = 3.388 \times 10^4$ kN/m^2, $k_{10} = 4.423 \times 10^4$ kN, $d_r = 282$ kN s/m^2, $d_{r1} = 93$ kN s, $\alpha = 1.0$ and $\xi = 0.1$. The k_0 and k_{10} are the standard stiffness of the foundation, while stiff ratio is defined as $\alpha = k/k_0 = k_1/k_{10}$. Likewise, the d_r and d_{r1} are the critical damps of the foundation and damping ratio is defined as $\xi = d/d_r = d_1/d_{r1}$. The corresponding dilatation and shear wave velocities are $c_\mathrm{p} = 2961$ m/s and $c_\mathrm{s} = 1745$ m/s, respectively. For the purpose of demonstrating the correctness and accuracy of the current truncated boundary condition, the system is considered under an in-plane square sine impulse disturbance (deflection or rotation) as follows:

$$f(T) = \begin{cases} \sin^2(\pi T/2) & 0 \leqslant T \leqslant 2.0 \\ 0.0 & T \geqslant 2.0 \end{cases} \tag{51}$$

where $T = c_\mathrm{s}t/h$ is dimensionless time, and h the height of beam.

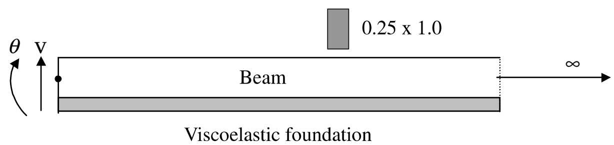


Fig. 3. Semi-infinite Timoshenko beam on viscoelastic foundation.

The time step in the analysis was chosen in such a way that the least error was obtained in approximating the loading function, also the time step must be less than the corresponding maximum value, which satisfies the requirement for stable integral conditions. In this particular example, the dimensionless time step used is 0.005. The total integrating time is 10.

The first case considered in this subsection is to investigate the convergence of the high-order accurate nonreflecting boundary condition. The number of residual radiation function in the following analysis is used as $n = 5$, 10 and 20, respectively. The input deflection or rotation at the boundary is prescribed by Eq. (51), i.e., when the deflection is given, the rotation is zero and vice versa. The resultant forces at the boundary of the semi-infinite beam are shown in Fig. 4 for various cases. Obviously the longer is the numerical simulation time, the more the auxiliary residual functions are required to obtain a high accurate solution. It can be seen that the numerical solution has excellent agreement with the exact solution in the case of $n = 20$. Nevertheless, this indicates that the present artificial boundary condition is a very accurate artificial boundary for solving elastic wave propagation problems in a foundation beam, if the order of the residual radiation function used is high enough in the numerical analysis.

Furthermore, the finite element method with the present high-order accurate boundary condition is used to solve the dynamic responses in semi-infinite foundation beam subjected to impulse loads. The finite element model for foundation beam is shown in Fig. 5, whose element size is $l = 1$ m. The material parameters are exactly the same as those in the first case. In order to examine the effect of location of the truncated boundary on the numerical results, four different region sizes, namely $L = 5$, 10, 15, and 30 m, are

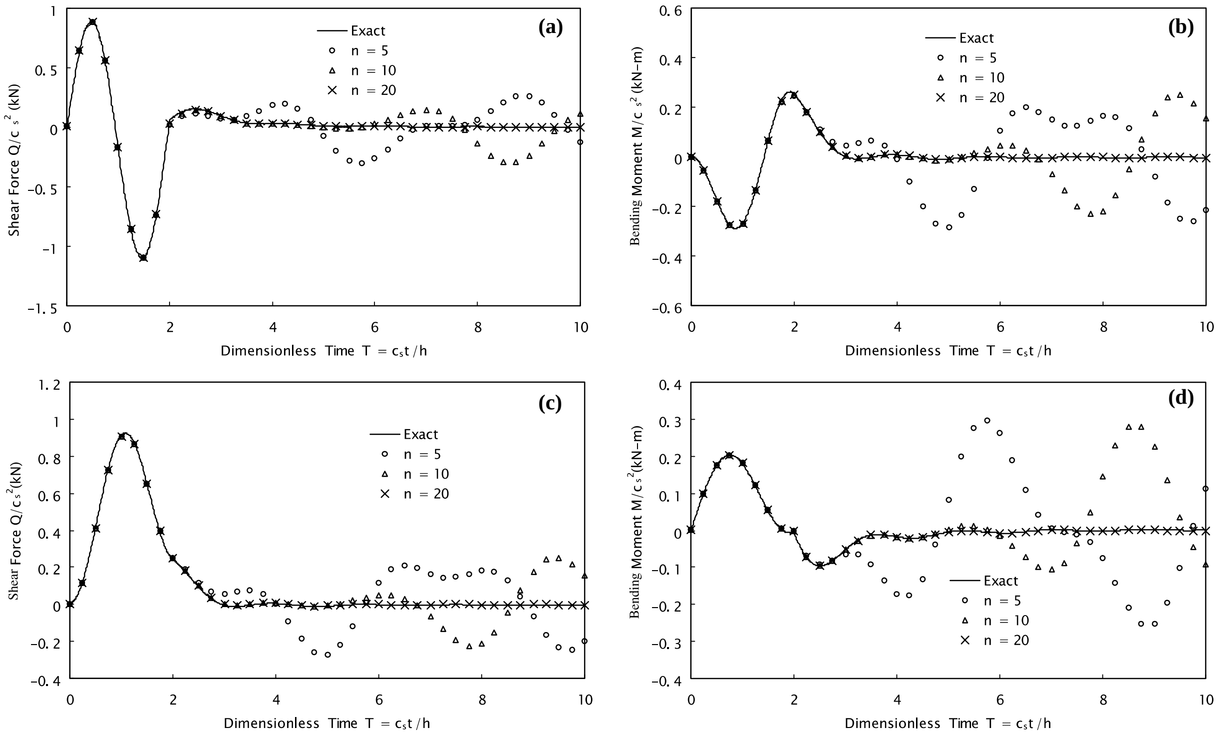


Fig. 4. Comparison of the numerical solutions with the exact solutions due to the different order of residual radiation functions ($n = 5, 10, 20$): (a) shear force at boundary due to lateral deflection at boundary; (b) bending moment at boundary due to lateral deflection at boundary; (c) shear force at boundary due to rotation at boundary; (d) bending moment at boundary due to rotation at boundary.

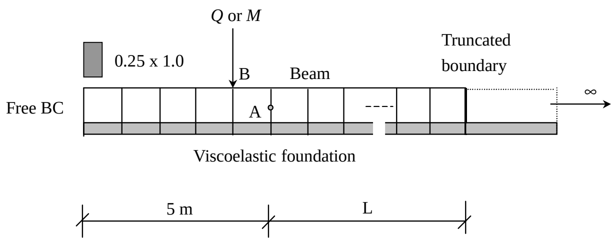


Fig. 5. Finite element model for a semi-infinite homogeneous Timoshenko beam on viscoelastic foundation.

considered in the finite element analysis. The foundation beam is subjected to an impulse loads described in Eq. (51). Fig. 6 shows the effect of the artificial boundary locations on the dynamic responses at the point A. The numerical solutions resulting from both the free boundary conditions and fixed boundary conditions on the truncated boundary are also shown in the corresponding figures (marked by (F) and (F′)). It is obvious that even though the size of beam modeled by the finite element is very small, a very accurate numerical solution can be obtained with the high-order accurate boundary conditions (marked by (H)). However, there exist significant differences between the results of the free boundary conditions and those of the high-order boundary conditions imposed on the truncated boundary, while there exists a consistency between the results of a large size of finite element model with the fixed boundary conditions and those of

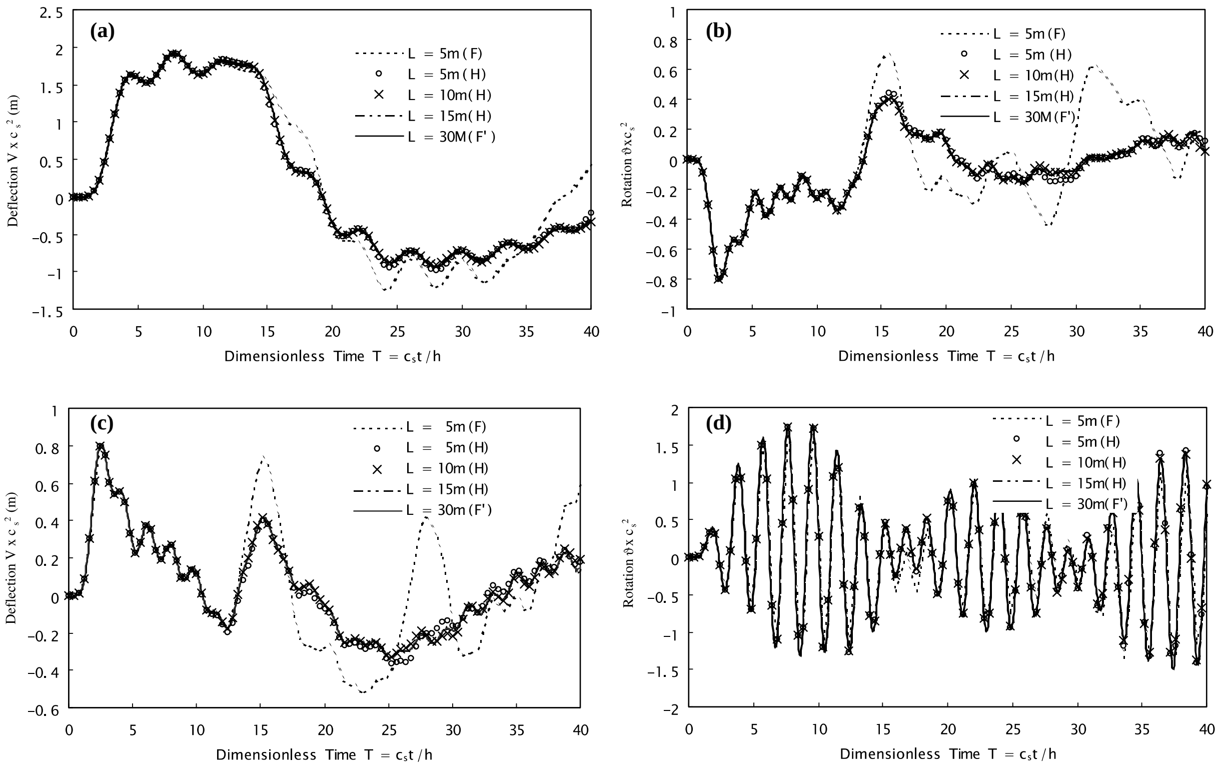


Fig. 6. Effect of truncated boundary location on the response of beam subjected to impulse load (F: free boundary conditions; F′: fixed boundary conditions; H: high-order accurate boundary conditions): (a) deflection at node A due to lateral force at node B; (b) rotation at node A due to lateral force at node B; (c) deflection at node A due to bending moment at node B; (d) rotation at node A due to bending moment at node B.

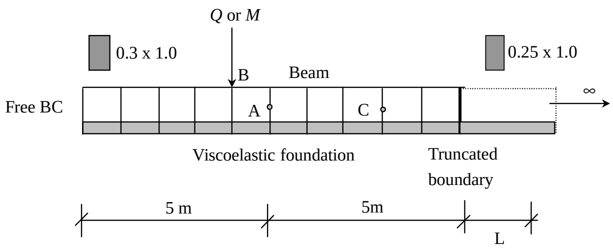


Fig. 7. Finite element model for a semi-infinite inhomogeneous Timoshenko beam on viscoelastic foundation.

the high-order boundary conditions imposed on the truncated boundary. This further indicates that the proposed nonreflecting artificial boundary is accurate enough to simulate the elastic wave propagation problem in infinite beam on viscoelastic foundation.

5.2. A partly inhomogeneous semi-infinite beam on viscoelastic foundation

A partly inhomogeneous semi-infinite beam with different thickness overlying on the viscoelastic foundation is shown in Fig. 7. The inhomogeneous character makes it difficulty to get an analytical solution

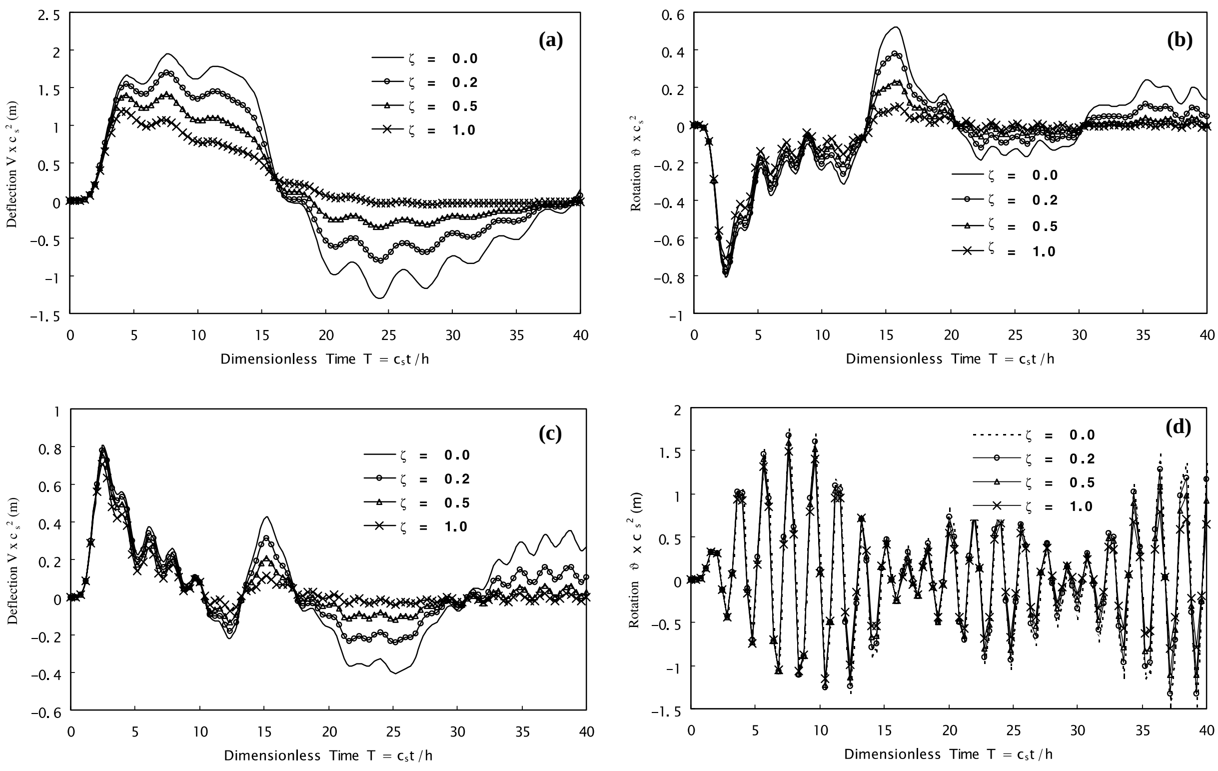


Fig. 8. Effects of viscous damping on responses of the beam subjected to an impulse load: (a) deflection at node A due to lateral force at node B; (b) rotation at node A due to lateral force at node B; (c) deflection at node A due to bending moment at node B; (d) rotation at node A due to bending moment at node B.

(Felszeghy, 1996a,b). However, there is no difference between the methods for the inhomogeneous foundation beam problem and the aforementioned homogeneous problem when using the current proposed numerical method. The truncated boundary is imposed on the bi-material interface of the inhomogeneous foundation beam. The finite region is modeled by finite element with size of 1 m, while the semi-infinite homogeneous part is represented by high-order accurate boundary condition. Parameters used in finite element model are: $A = 0.3 \times 1.0$ m^2, $I = 0.025$ m^4, $E = 4.29 \times 10^5$ kN m^2, $\gamma = 0.2$, $\rho = 2.35 \times 10^3$ kg/m^3, $k_0 = 4.066 \times 10^4$ kN/m^2, $k_{10} = 5.307 \times 10^4$ kN, $d_r = 339$ kN s/m^2, $d_{r1} = 112$ kN s, $\alpha = 1.0$, those for the semi-infinite homogeneous foundation beam are the same as in the first example.

The first case considered in this subsection is to examine the effect of the viscous damping of foundation on the numerical results. Four damping values, $\xi = 0.0, 0.2, 0.5$ and 1.0, are considered in the finite element analysis. The system is still subjected to the impulse load described in Eq. (51). The time step and the integrating time used are the same as those in the previous numerical example. Fig. 8 shows the effect of the viscous damping on the transient responses of the inhomogeneous semi-infinite foundation beam. It is clear that the damping obviously affects the transient response of the system. This does demonstrate that the current high-order nonreflecting artificial boundary condition is very effective for solving elastic wave propagation problems in the inhomogeneous infinite beam on viscoelastic foundation.

Alternatively, with a fixed viscous damping ($\xi = 0.2$), four values of $\alpha = 0.1, 1.0, 10, 50$ are used to examine the effect of the stiffness of foundation on the numerical results. Fig. 9 shows the effect of the stiffness of foundation on the transient responses of the inhomogeneous semi-infinite foundation beam. Clearly, the

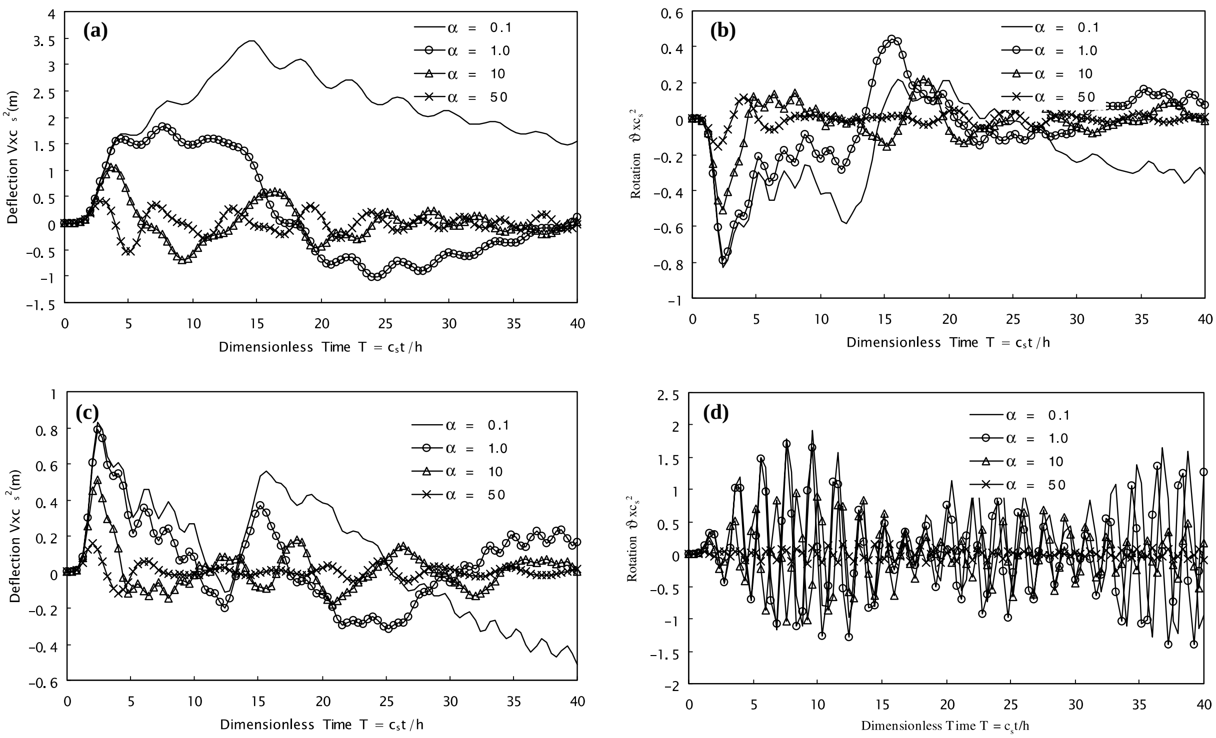


Fig. 9. Effects of foundation stiffness on responses of the beam subjected to an impulse load: (a) deflection at node A due to lateral force at node B; (b) rotation at node A due to lateral force at node B; (c) deflection at node A due to bending moment at node B; (d) rotation at node A due to bending moment at node B.

related numerical results show that the foundation stiffness have significant effects on the transient responses of the semi-infinite foundation beam, in particular, the soft foundation case.

Finally, the inhomogeneous semi-infinite foundation beam subjected to a resonant load is considered to obtain the long time response of the system. The following resonant load is used in the corresponding numerical analysis:

$$f(T) = \sin(\pi T) \quad 0 \leqslant T \leqslant 40 \tag{52}$$

Except for $\alpha = 0.1$ and $\xi = 0.2$, the other material parameters are exactly the same as those in the previous cases of this subsection. The time step and the integrating time used are the same as those in the previous numerical example. Numerical results at different locations are compared with the results, which are obtained by a large size of foundation beam mode with, $L = 25$ m and the fixed truncated boundary condition (marked by (C)). Fig. 10 shows the corresponding responses at locations A and C of the foundation beam. It can be observed that even for the resonant load case, the numerical results from the proposed method still agree well with those by a large conventional finite element model with Dirichlet boundary conditions. This demonstrates the usefulness and accuracy of the present high-order accuracy boundary conditions in dealing with elastic wave propagation problems in semi-infinite foundation beam.

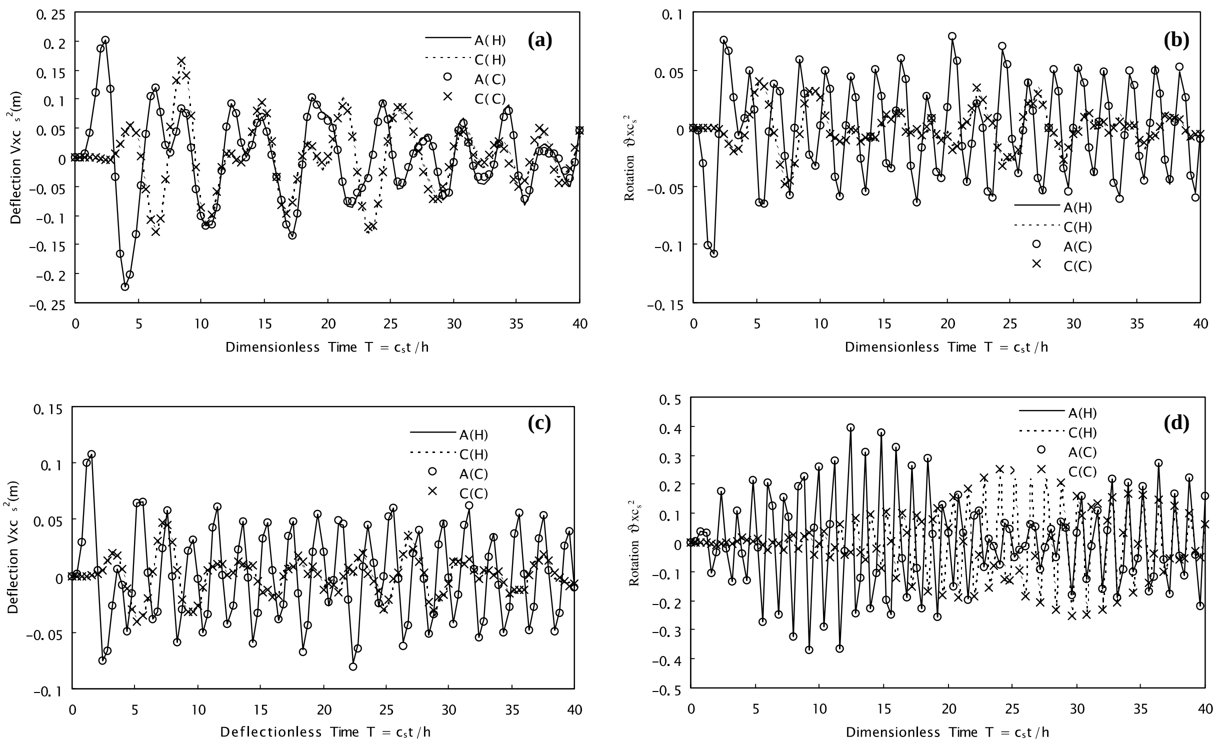


Fig. 10. Long time responses of the foundation beam subjected to a resonant load (H: high-order accurate boundary conditions; C: conventional finite element method): (a) deflections at node A and C due to lateral resonant force at node B; (b) rotations at node A and C due to lateral resonant force at node B; (c) deflections at node A and C due to resonant bending moment at node B; (d) rotation at node A and C due to resonant bending moment at node B.

6. Conclusions

In this paper, a high-order nonreflecting artificial boundary condition is proposed to simulate the transient elastic wave propagation in an infinite beam on viscoelastic foundation. Both the operator splitting technique and residual radiation function are used to derive the high-order accurate artificial boundary conditions in a rigorously mathematical manner. Those complicated convolution calculations commonly used in other analytical and numerical methods can be avoided in the current artificial boundary condition due to neither the fundamental solution nor the impulse response function is needed.

The proposed high-order accurate artificial boundary condition is local in time for the elastic wave problems in foundation beam. It can be combined easily with finite element method to deal with elastic wave propagation problems in infinite beam including viscoelastic foundation, because the context of the artificial boundary condition is the numerical method. The performance of the proposed high-order nonreflecting artificial boundary is illustrated by some transient elastic wave propagation problems in the semi-infinite beam with inclusion of viscoelastic foundation. The results support that the high-order nonreflecting artificial boundary condition proposed in this paper is accurate for solving elastic wave propagation problems of infinite (or semi-infinite) beams on viscoelastic foundation.

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